

# On reciprocity formula of character Dedekind sums and the integral of products of Bernoulli polynomials

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## Abstract

We give a simple proof for the reciprocity formulas of character Dedekind sums associated with two primitive characters, whose modulus need not to be same, by utilizing the character analogue of the Euler–MacLaurin summation formula. Moreover, we extend known results on the integral of products of Bernoulli polynomials by considering the integral

$$\int_0^x B_{n_1}(b_1 z + y_1) \cdots B_{n_r}(b_r z + y_r) dz,$$

where  $b_l$  ( $b_l \neq 0$ ) and  $y_l$  ( $1 \leq l \leq r$ ) are real numbers. As a consequence of this integral we establish a connection between the reciprocity relations of sums of products of Bernoulli polynomials and of the Dedekind sums.

**Keywords:** Dedekind sums, Bernoulli polynomials, Bernoulli numbers, Euler–MacLaurin formula, Laplace transform.

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## 1 Introduction

Let

$$((x)) = \begin{cases} x - [x] - 1/2, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$$

with  $[x]$  being the largest integer  $\leq x$ . For positive integers  $c$  and integers  $b$  the classical Dedekind sum  $s(b, c)$ , arising in the theory of Dedekind  $\eta$ -function, was introduced by R. Dedekind in 1892 as

$$s(b, c) = \sum_{j(\bmod c)} \left( \left( \frac{j}{c} \right) \right) \left( \left( \frac{bj}{c} \right) \right).$$

The most important property of Dedekind sums is the reciprocity theorem

$$s(b, c) + s(c, b) = -\frac{1}{4} + \frac{1}{12} \left( \frac{b}{c} + \frac{c}{b} + \frac{1}{bc} \right) \quad (1)$$

for  $\gcd(b, c) = 1$ . For several proofs of (1) see [22] (for recent studies on Dedekind sums the reader may consult to [12, 13, 15, 16, 24]). These sums were later generalized by various mathematicians and the corresponding

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reciprocity laws were obtained. Apostol [2] generalized  $s(b, c)$  by defining

$$s_p(b, c) = \sum_{j=0}^{c-1} \overline{B}_p\left(\frac{bj}{c}\right) \overline{B}_1\left(\frac{j}{c}\right),$$

where  $\overline{B}_p(x)$  is the  $p$ th Bernoulli function defined by

$$\overline{B}_n(x) = B_n(x - [x]) \text{ for } n > 1 \text{ and } \overline{B}_1(x) = ((x)).$$

Here  $B_n(x)$  denotes the  $n$ th Bernoulli polynomial (see Section 2). The sum  $s_p(b, c)$  satisfies the following reciprocity formula for odd  $p$  and  $\gcd(b, c) = 1$  [2]:

$$(p+1) \left( bc^p s_p(b, c) + cb^p s_p(c, b) \right) = \sum_{j=0}^{p+1} \binom{p+1}{j} (-1)^j b^j c^{p+1-j} B_{p+1-j} B_j + p B_{p+1}. \quad (2)$$

The another generalization is due to Berndt [3]. He gave a character transformation formula similar to those for the Dedekind  $\eta$ -function and defined Dedekind sums with character  $s(b, c : \chi)$  by

$$s(b, c : \chi) = \sum_{n=0}^{ck-1} \chi(n) \overline{B}_{1,\chi}\left(\frac{bn}{c}\right) \overline{B}_1\left(\frac{n}{ck}\right),$$

where  $\chi$  denotes a non-principal primitive character modulo  $k$  and  $\overline{B}_{p,\chi}(x)$  is the  $p$ th generalized Bernoulli function, with  $\overline{B}_{p,\chi}(0) = B_{p,\chi}$  (see (8)). Using character transformation formula, he also derived the following reciprocity law [3, Theorem 4]

$$s(c, b : \chi) + s(b, c : \overline{\chi}) = B_{1,\chi} B_{1,\overline{\chi}}, \quad (3)$$

whenever  $b$  and  $c$  are coprime positive integers, and either  $c$  or  $b \equiv 0 \pmod{k}$ . For the proofs of (1) and (3) via (periodic) Poisson summation formula see [5]. The sum  $s(b, c : \chi)$  is generalized by Cenkei et al [9] as

$$s_p(b, c : \chi) = \sum_{n=0}^{ck-1} \chi(n) \overline{B}_{p,\chi}\left(\frac{bn}{c}\right) \overline{B}_1\left(\frac{n}{ck}\right) \quad (4)$$

and the following reciprocity formula is established:

Let  $p$  be odd and  $b, c$  be coprime positive integers. Let  $\chi$  be a non-principal primitive character of modulus  $k$ , where  $k$  is a prime number if  $\gcd(k, bc) = 1$ , otherwise  $k$  is an arbitrary integer. Then

$$\begin{aligned} & (p+1) (bc^p s_p(b, c : \chi) + cb^p s_p(c, b : \overline{\chi})) \\ &= \sum_{j=0}^{p+1} \binom{p+1}{j} b^j c^{p+1-j} B_{j,\overline{\chi}} B_{p+1-j,\chi} + \frac{p}{k} \chi(c) \overline{\chi}(-b) (k^{p+1} - 1) B_{p+1}. \end{aligned} \quad (5)$$

Recently, the authors [7] have used the character analogue of the Euler–MacLaurin summation formula in order to prove (5) (this method was also exploited in [6, 10]). We perceive from this formula that the sum  $s_p(b, c : \chi)$  can be generalized to sums such as  $s_p(b, c : \chi_1, \chi_2)$  involving two primitive characters.

In this paper, we systematically generalize  $s_p(b, c : \chi)$  to sums involving two primitive characters and prove the corresponding reciprocity formulas.

Firstly, we define character Dedekind sum involving primitive characters  $\chi_1$  and  $\chi_2$  of modulus  $k$  by

$$s_p(b, c : \chi_1, \chi_2) = \sum_{n=0}^{ck-1} \chi_1(n) \overline{B}_{p,\chi_2}\left(\frac{bn}{c}\right) \overline{B}_1\left(\frac{n}{ck}\right),$$

which is a natural generalization of the sum  $s_p(b, c : \chi)$  given by (4), i.e.,  $s_p(b, c : \chi, \chi) = s_p(b, c : \chi)$ . Once again utilizing the power of the Euler–MacLaurin summation formula we derive the following reciprocity formula.

**Theorem 1** *Let  $b, c$  be positive integers with  $q = \gcd(b, c)$  and  $p > 1$ . Let  $\chi_1$  and  $\chi_2$  be non-principal primitive characters of modulus  $k$ . For  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = 1$  the following reciprocity formula holds*

$$(p+1) (bc^p s_p(b, c : \chi_1, \chi_2) + cb^p s_p(b, c : \overline{\chi_2}, \overline{\chi_1})) \\ = \sum_{j=0}^{p+1} \binom{p+1}{j} b^j c^{p+1-j} B_{j, \overline{\chi_1}} B_{p+1-j, \chi_2} + pq^{p+1} k^{p-1} \sum_{h=1}^{k-1} \sum_{a=1}^{k-1} \chi_1(h) \overline{\chi_2}(a) \overline{B}_{p+1} \left( \frac{ca + bh}{qk} \right).$$

Secondly, we define the sum  $\widehat{S}_p(b, c : \chi_1, \chi_2)$  for primitive characters  $\chi_1$  and  $\chi_2$  having modulus  $k_1$  and  $k_2$ , need not to be same, by

$$\widehat{S}_p(b, c : \chi_1, \chi_2) = \sum_{n=0}^{ck_1 k_2 - 1} \chi_1(n) \overline{B}_{p, \chi_2} \left( \frac{nb}{c} \right) \overline{B}_1 \left( \frac{n}{ck_1 k_2} \right),$$

which reduces to  $s_p(b, c : \chi_1, \chi_2)$  for  $k_1 = k_2$ .

Finally, for primitive characters  $\chi_1$  and  $\chi_2$  having modulus  $k_1$  and  $k_2$ , we consider the following sum

$$\widetilde{S}_p(b, c : \chi_1, \chi_2) = \sum_{n=0}^{ck_1 - 1} \chi_1(n) \overline{B}_{p, \chi_2} \left( \frac{nbk_2}{ck_1} \right) \overline{B}_1 \left( \frac{n}{ck_1} \right),$$

which generalizes the previous sums. More clearly,  $\widetilde{S}_p(b, c : \chi_1, \chi_2) = s_p(b, c : \chi_1, \chi_2)$  for  $k_1 = k_2$ , and  $\widetilde{S}_p(bk_1, ck_2 : \chi_1, \chi_2) = \widehat{S}_p(b, c : \chi_1, \chi_2)$ . We obtain the following reciprocity formula for  $\widetilde{S}_p(b, c : \chi_1, \chi_2)$ .

**Theorem 2** *Let  $b, c$  be positive integers with  $q = \gcd(b, c)$  and  $p > 1$ . Let  $\chi_1$  and  $\chi_2$  be non-principal primitive characters of modulus  $k_1$  and  $k_2$ , respectively. For  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = 1$  the following reciprocity formula holds*

$$(p+1) \left( bk_2 (ck_1)^p \widetilde{S}_p(b, c : \chi_1, \chi_2) + ck_1 (bk_2)^p \widetilde{S}_p(c, b : \overline{\chi_2}, \overline{\chi_1}) \right) \\ = \sum_{j=0}^{p+1} \binom{p+1}{j} (bk_2)^j (ck_1)^{p+1-j} B_{j, \overline{\chi_1}} B_{p+1-j, \chi_2} \\ + pq^{p+1} (k_1 k_2)^p \sum_{h=1}^{k_1} \sum_{j=1}^{k_2} \chi_1(h) \overline{\chi_2}(j) \overline{B}_{p+1} \left( \frac{cj}{qk_2} + \frac{bh}{qk_1} \right).$$

In particular we have the following corollary.

**Corollary 3** *Let  $b, c$  be positive integers with  $q = \gcd(b, c)$  and  $p > 1$ . Let  $\chi_1$  and  $\chi_2$  be non-principal primitive characters of modulus  $k_1$  and  $k_2$ , respectively. For  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = 1$  and  $k_1 \neq k_2$ , the following reciprocity formula holds*

$$(p+1) \left( bc^p \widehat{S}_p(b, c : \overline{\chi_1}, \chi_2) + cb^p \widehat{S}_p(c, b : \overline{\chi_2}, \chi_1) \right) = \sum_{j=0}^{p+1} \binom{p+1}{j} c^j b^{p+1-j} B_{p+1-j, \chi_1} B_{j, \chi_2}.$$

Moreover, we derive an explicit formula for the following type integral

$$\int_0^x B_{n_1}(b_1 z + y_1) \cdots B_{n_r}(b_r z + y_r) dz$$

(see Proposition 11). Since this theorem is also valid for the Appell polynomials, the earlier results given by Liu et al [17], Hu et al [14], and Agoh and Dilcher [1] are direct consequences of the derived formula (see Remark 10). As a consequence of this integral, we have the following reciprocity relation for Bernoulli polynomials, which generalizes [1, Proposition 2].

**Corollary 4** For all  $n, m \geq 0$  we have

$$\begin{aligned}
& \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} B_{n-a}(b_1 x + y_1) B_{m+a+1}(b_2 x + y_2) \\
& - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} B_{m-a}(b_2 x + y_2) B_{n+a+1}(b_1 x + y_1) \\
& = \frac{(-1)^{m+1}}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^a \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} B_{m+n+1-a}(y_1) B_a(y_2), \tag{6}
\end{aligned}$$

where  $b_l$  ( $b_l \neq 0$ ) and  $y_l$  ( $1 \leq l \leq r$ ) are real numbers.

**Remark 5** 1) If  $(n+m)$  is even and  $y_1, y_2$  are in  $\{0, 1/2, 1\}$ , then the right-hand side of (6) reduces to

$$(-1)^n \frac{m+n+1}{b_1} \left\{ \left( \frac{b_2}{b_1} \right)^{m-1} B_{m+n}(y_1) B_1(y_2) - \left( \frac{b_1}{b_2} \right)^n B_1(y_1) B_{m+n}(y_2) \right\}.$$

2) If  $(n+m)$  is odd and  $y_1 = y_2 = 0$  or  $1$ , then the right-hand side of (6) is closely related to the reciprocity formula of Apostol's Dedekind sums given by (2). In fact, we have for all  $x$  and odd integer  $p = m+n$ , ( $n, m \geq 0$ )

$$\begin{aligned}
& (p+1) (b_1 b_2^p s_p(b_1, b_2) + b_2 b_1^p s_p(b_2, b_1)) \\
& = \sum_{a=0}^n (-1)^{n-a} \binom{m+n+1}{n-a} b_1^{m+a+1} b_2^{n-a} B_{n-a}(b_1 x) B_{m+a+1}(b_2 x) \\
& + \sum_{a=0}^m (-1)^{m-a} \binom{m+n+1}{m-a} b_2^{n+a+1} b_1^{m-a} B_{m-a}(b_2 x) B_{n+a+1}(b_1 x) + q^{p+1} p B_{p+1} \\
& = \sum_{a=0}^{m+n+1} (-1)^a \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} B_{m+n+1-a} B_a + q^{p+1} p B_{p+1},
\end{aligned}$$

where  $q = \gcd(b_1, b_2)$ .

We further derive the Laplace transform of  $\overline{B}_n(tu+y)$  (see (38)), which coincides with [19, Lemma 4] for  $t=1$  and  $y \in \mathbb{Z}$ . Similar results are also obtained for the generalized Bernoulli polynomials.

We summarize this study as follows: Section 2 is the preliminary section where we give definitions and known results needed. In Section 3, we prove the reciprocity formulas for character Dedekind sums by using the character analogue of the Euler–MacLaurin summation formula. In Section 4, we first derive a formula for the integral having more general integrands. By this, we extend known results on the integral of products of Appell polynomials and illustrate a formula for the integral of products of Bernoulli polynomials. Furthermore, the proof of Corollary 4 and the Laplace transform of  $\overline{B}_n(tu+y)$  are given.

## 2 Preliminaries

The Bernoulli polynomials  $B_n(x)$  are defined by means of the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi) \tag{7}$$

and  $B_n = B_n(0)$  are the Bernoulli numbers with  $B_0 = 1$ ,  $B_1 = -1/2$  and  $B_{2n+1} = B_{2n-1}(1/2) = 0$  for  $n \geq 1$ .

Let  $\chi$  be a primitive character of modulus  $k$ . The generalized Bernoulli polynomials  $B_{n,\chi}(x)$  are defined by means of the generating function [4]

$$\sum_{a=0}^{k-1} \frac{\bar{\chi}(a) t e^{(a+x)t}}{e^{kt} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!} \quad (|t| < 2\pi/k)$$

and  $B_{n,\chi} = B_{n,\chi}(0)$  are the generalized Bernoulli numbers. In particular, if  $\chi_0$  is the principal character, then  $B_{n,\chi_0}(x) = B_n(x)$  for  $n \geq 0$  and  $B_{0,\chi}(x) = 0$  for  $\chi \neq \chi_0$ . The generalized Bernoulli functions  $\bar{B}_{n,\chi}(x)$ , are functions with period  $k$ , may be defined by ([4, Theorem 3.1])

$$\bar{B}_{m,\chi}(x) = k^{m-1} \sum_{n=1}^{k-1} \bar{\chi}(n) \bar{B}_m\left(\frac{n+x}{k}\right), \quad m \geq 1, \quad (8)$$

for all real  $x$ . We recall some properties that we need in the sequel.

$$\frac{d}{dx} B_m(x) = m B_{m-1}(x) \quad \text{and} \quad \frac{d}{dx} B_{m,\chi}(x) = m B_{m-1,\chi}(x), \quad m \geq 1, \quad (9)$$

$$\frac{d}{dx} \bar{B}_{m,\chi}(x) = m \bar{B}_{m-1,\chi}(x), \quad m \geq 2 \quad \text{and} \quad \bar{B}_{m,\chi}(k) = \bar{B}_{m,\chi}(0) = B_{m,\chi}, \quad (10)$$

$$B_{m,\chi}(-x) = (-1)^m \chi(-1) B_{m,\chi}(x), \quad m \geq 0. \quad (11)$$

It is also known that the degree of Bernoulli polynomials  $B_n(x)$  is  $n$ , and the degree of generalized Bernoulli polynomials  $B_{n,\chi}(x)$  is not greater than  $n-1$ .

We also need the character analogue of the Euler–MacLaurin summation formula, due to Berndt [4], which is presented here in the following form.

**Theorem 6** ([4, Theorem 4.1]) *Let  $f \in C^{(l+1)}[\alpha, \beta]$ ,  $-\infty < \alpha < \beta < \infty$ . Then,*

$$\begin{aligned} \sum_{\alpha \leq n \leq \beta} ' \chi(n) f(n) &= \chi(-1) \sum_{j=0}^l \frac{(-1)^{j+1}}{(j+1)!} \left( \bar{B}_{j+1,\bar{\chi}}(\beta) f^{(j)}(\beta) - \bar{B}_{j+1,\bar{\chi}}(\alpha) f^{(j)}(\alpha) \right) \\ &\quad + \chi(-1) \frac{(-1)^l}{(l+1)!} \int_{\alpha}^{\beta} \bar{B}_{l+1,\bar{\chi}}(u) f^{(l+1)}(u) du. \end{aligned}$$

where the dash indicates that if  $n = \alpha$  or  $n = \beta$ , then only  $\frac{1}{2}\chi(\alpha)f(\alpha)$  or  $\frac{1}{2}\chi(\beta)f(\beta)$  is counted, respectively.

### 3 Proofs of Theorems 1 and 2

We recall the sum  $s_p(b, c : \chi_1, \chi_2)$  defined by

$$s_p(b, c : \chi_1, \chi_2) = \sum_{n=0}^{ck-1} \chi_1(n) \bar{B}_{p,\chi_2}\left(\frac{bn}{c}\right) \bar{B}_1\left(\frac{n}{ck}\right),$$

where  $\chi_1$  and  $\chi_2$  are primitive characters of modulus  $k$ . In view of (11) it is easy to see that

$$s_p(b, c : \chi_1, \chi_2) = (-1)^{p+1} \chi_1(-1) \chi_2(-1) s_p(b, c : \chi_1, \chi_2), \quad (12)$$

which entails  $s_p(b, c : \chi_1, \chi_2) = 0$  for  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = -1$ .

**Proof of Theorem 1.** With the aid of  $\overline{B}_1(x) = x - 1/2$  for  $0 < x < 1$ , we have

$$cks_p(b, c; \chi_1, \chi_2) = \sum_{n=0}^{ck-1} \chi_1(n) n \overline{B}_{p, \chi_2} \left( \frac{bn}{c} \right) - \frac{ck}{2} \sum_{n=0}^{ck-1} \chi_1(n) \overline{B}_{p, \chi_2} \left( \frac{bn}{c} \right). \quad (13)$$

Thus, let  $f(x) = x \overline{B}_{p, \chi_2}(xb/c)$ ,  $\alpha = 0$  and  $\beta = ck$  in Theorem 6 and let  $p > 1$ . Equation (10) entails that  $f \in C^{(p-1)}[\alpha, \beta]$  and

$$\frac{d^j}{dx^j} f(x) = \frac{p!}{(p-j)!} \left( \frac{b}{c} \right)^j x \overline{B}_{p-j, \chi_2} \left( \frac{b}{c} x \right) + j \frac{p!}{(p+1-j)!} \left( \frac{b}{c} \right)^{j-1} \overline{B}_{p+1-j, \chi_2} \left( \frac{b}{c} x \right) \quad (14)$$

for  $0 \leq j \leq p-1$ . Therefore with the help of (10) and Theorem 6 we have

$$\begin{aligned} \sum_{n=1}^{ck-1} \chi_1(n) n \overline{B}_{p, \chi_2} \left( \frac{bn}{c} \right) &= \chi_1(-1) \frac{ck}{p+1} \sum_{j=0}^l (-1)^{j+1} \binom{p+1}{j+1} \left( \frac{b}{c} \right)^j B_{j+1, \overline{\chi_1}} B_{p-j, \chi_2} \\ &\quad + \chi_1(-1) \binom{p}{l+1} \left( -\frac{b}{c} \right)^l bc \int_0^k x \overline{B}_{l+1, \overline{\chi_1}}(cx) \overline{B}_{p-(l+1), \chi_2}(bx) dx \\ &\quad + \chi_1(-1) (-1)^l \binom{p}{l} \left( \frac{b}{c} \right)^l c \int_0^k \overline{B}_{l+1, \overline{\chi_1}}(cx) \overline{B}_{p-l, \chi_2}(bx) dx \end{aligned} \quad (15)$$

for  $1 \leq l+1 \leq p-1$ . On the other hand, applying character analogue of the Euler–MacLaurin summation formula to the generalized Bernoulli function  $\overline{B}_{p+1, \chi_2}(bx/c)$  gives

$$\sum_{n=1}^{ck-1} \chi_1(n) \overline{B}_{p+1, \chi_2} \left( \frac{bn}{c} \right) = \chi_1(-1) (-1)^l \binom{p+1}{l+1} \left( \frac{b}{c} \right)^{l+1} c \int_0^k \overline{B}_{l+1, \overline{\chi_1}}(cx) \overline{B}_{p-l, \chi_2}(bx) dx. \quad (16)$$

Let  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = 1$ . Taking into account that

$$\sum_{n=1}^{ck-1} \chi_1(n) \overline{B}_{p, \chi_2} \left( \frac{bn}{c} \right) = 0 \quad (17)$$

for  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = 1$ , it follows from (13), (15) and (16) that

$$\begin{aligned} cks_p(b, c; \chi_1, \chi_2) &= \chi_1(-1) \frac{ck}{p+1} \sum_{j=0}^l (-1)^{j+1} \binom{p+1}{j+1} \left( \frac{b}{c} \right)^j B_{j+1, \overline{\chi_1}} B_{p-j, \chi_2} \\ &\quad + \chi_1(-1) \binom{p}{l+1} \left( -\frac{b}{c} \right)^l bc \int_0^k x \overline{B}_{l+1, \overline{\chi_1}}(cx) \overline{B}_{p-(l+1), \chi_2}(bx) dx \\ &\quad + \frac{l+1}{p+1} \frac{c}{b} \sum_{n=1}^{ck-1} \chi_1(n) \overline{B}_{p+1, \chi_2} \left( \frac{bn}{c} \right). \end{aligned} \quad (18)$$

We first set  $l + 1 = p - 1$  in (18) and then multiply by  $bc^{p-1}/k$ . So we have

$$\begin{aligned} bc^p s_p(b, c : \chi_1, \chi_2) &= \frac{\chi_1(-1)}{p+1} \sum_{j=1}^{p-1} (-1)^j \binom{p+1}{j} b^j c^{p+1-j} B_{j, \overline{\chi_1}} B_{p+1-j, \chi_2} \\ &\quad + (-1)^p \chi_1(-1) \frac{pb^p c^2}{k} \int_0^k x \overline{B}_{p-1, \overline{\chi_1}}(cx) \overline{B}_{1, \chi_2}(bx) dx \\ &\quad + \frac{p-1}{p+1} \frac{c^p}{k} \sum_{n=1}^{ck-1} \chi_1(n) \overline{B}_{p+1, \chi_2} \left( \frac{bn}{c} \right). \end{aligned} \quad (19)$$

Now we first set  $l = 0$ , interchange  $b$  and  $c$  and also  $\chi_1$  and  $\overline{\chi_2}$  in (18), and then multiply by  $cb^{p-1}/k$ . Thus we have

$$\begin{aligned} cb^p s_p(c, b : \overline{\chi_2}, \overline{\chi_1}) &= -\chi_2(-1) cb^p B_{p, \overline{\chi_1}} B_{1, \chi_2} \\ &\quad + \overline{\chi_2}(-1) \frac{pb^p c^2}{k} \int_0^k x \overline{B}_{p-1, \overline{\chi_1}}(cx) \overline{B}_{1, \chi_2}(bx) dx \\ &\quad + \frac{1}{p+1} \frac{b^p}{k} \sum_{n=1}^{bk-1} \overline{\chi_2}(n) \overline{B}_{p+1, \overline{\chi_1}} \left( \frac{cn}{b} \right). \end{aligned} \quad (20)$$

We now consider the sum

$$\sum_{n=1}^{ck-1} \chi_1(n) \overline{B}_{p+1, \chi_2} \left( \frac{bn}{c} \right).$$

As mentioned in (17) the sum vanishes when  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = -1$ . For  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = 1$  and  $\gcd(b, c) = 1$ , first setting  $n = h + mk$ , where  $1 \leq h \leq k-1$ ,  $0 \leq m \leq c-1$ , and then using (8) and Raabe formula

$$\sum_{m=0}^{c-1} \overline{B}_{p+1} \left( \frac{m+x}{c} \right) = c^{-p} \overline{B}_{p+1}(x),$$

we deduce that

$$\sum_{n=1}^{ck-1} \chi_1(n) \overline{B}_{p+1, \chi_2} \left( \frac{bn}{c} \right) = \left( \frac{k}{c} \right)^p \sum_{h=1}^{k-1} \sum_{j=1}^{k-1} \chi_1(h) \overline{\chi_2}(j) \overline{B}_{p+1} \left( \frac{cj}{k} + \frac{bh}{k} \right). \quad (21)$$

We mention also that if  $\gcd(bq, cq) = q$ , then

$$\sum_{n=1}^{qck-1} \chi_1(n) \overline{B}_{p+1, \chi_2} \left( \frac{qbn}{qc} \right) = q \sum_{n=1}^{ck-1} \chi_1(n) \overline{B}_{p+1, \chi_2} \left( \frac{bn}{c} \right).$$

Therefore, combining (19) and (20), with the use of  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = 1$ , (11) and (21), we obtain the reciprocity formula. ■

**Remark 7** The sum occurs in (21) was evaluated in [9, Lemma 5.5] when  $\chi_1 = \chi_2$ . In that case, the right-hand side of (21) becomes  $c^{1-p} \chi(c) \overline{\chi}(-b) (k^p - 1) \overline{B}_p(0)$  under the condition that  $k$  is a prime number when

$\gcd(k, bc) = 1$  and  $p$  is even, otherwise  $k$  is an arbitrary integer. By similar method, this sum can be evaluated when  $\chi_1 \neq \chi_2$ , in which case we have

$$\sum_{n=1}^{ck-1} \chi_1(n) \overline{B}_{p, \chi_2} \left( \frac{bn}{c} \right) = -\epsilon \frac{p!}{(2\pi i)^p} \left( \frac{k}{c} \right)^{p-1} G(b, \chi_1) G(c, \overline{\chi_2}) L(p, \overline{\chi_1} \chi_2)$$

for  $\gcd(b, c) = 1$  with  $c > 0$ . Here  $\epsilon = 1 + (-1)^p \chi_1(-1) \chi_2(-1)$  and  $\overline{\chi_1} \chi_2$  is a Dirichlet character modulo  $k$ , and  $G(b, \chi)$  and  $L(p, \overline{\chi_1} \chi_2)$  stands for the Gauss sum and the Dirichlet  $L$ -function, respectively. However, the statement of reciprocity formula must be given separately for the cases  $\chi_1 = \chi_2$  and  $\chi_1 \neq \chi_2$ .

**Proof of Theorem 2.** We first note that similar to  $s_p(b, c; \chi_1, \chi_2)$  the sum  $\tilde{S}_p(b, c; \chi_1, \chi_2)$  vanishes when  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = -1$ . Let  $\gcd(b, c) = 1$  with  $c > 0$ . Then the counterpart of (21) becomes

$$\sum_{n=1}^{ck_1} \chi_1(n) \overline{B}_{p+1, \chi_2} \left( \frac{nbk_2}{ck_1} \right) = \left( \frac{k_2}{c} \right)^p \sum_{h=1}^{k_1} \sum_{j=1}^{k_2} \chi_1(h) \overline{\chi_2}(j) \overline{B}_{p+1} \left( \frac{cj}{k_2} + \frac{bh}{k_1} \right) \quad (22)$$

when  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = 1$ , and this sum vanishes when  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = -1$ . Now let  $f(x) = x \overline{B}_{p, \chi_2}(xbk_2/ck_1)$ ,  $\alpha = 0$ ,  $\beta = ck$  in Theorem 6 and  $p > 1$ . Then adopting the arguments in the proof of Theorem 1, and using (22) rather than (21), the desired result follows. ■

### 3.1 Further consequences

Let  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = 1$ . If we take  $c = 1$  and  $b = k$  in (16). Using (10) and then the fact  $\sum_{n=1}^{k-1} \chi(n) = 0$ , we find that

$$\int_0^k \overline{B}_{l+1, \overline{\chi_1}}(x) \overline{B}_{p-l, \chi_2}(kx) dx = 0.$$

It is also obvious from (16), for  $b = c = 1$ ,

$$\chi_1(-1) (-1)^{l+1} \binom{p+1}{l+1} \int_0^k \overline{B}_{l+1, \overline{\chi_1}}(x) \overline{B}_{p-l, \chi_2}(x) dx = \sum_{n=1}^k \chi_1(n) \overline{B}_{p+1, \chi_2}(n).$$

Now let  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = -1$ . Then it follows from (16) and (17) that

$$\int_0^k \overline{B}_{l+1, \overline{\chi_1}}(cx) \overline{B}_{p-l, \chi_2}(bx) dx = 0. \quad (23)$$

It is seen from (12) and (13) that

$$\sum_{n=1}^{ck-1} \chi_1(n) n \overline{B}_{p, \chi_2} \left( \frac{bn}{c} \right) = \frac{ck}{2} \sum_{n=1}^{ck-1} \chi_1(n) \overline{B}_{p, \chi_2} \left( \frac{bn}{c} \right).$$

Since  $B_{p+1-j, \chi_2} = (-1)^{p+1-j} \chi_2(-1) B_{p+1-j, \chi_2}$  from (11), we conclude from (15), (21) and (23) that

$$\begin{aligned} & \binom{p}{l+1} \left( -\frac{b}{c} \right)^l b \int_0^k x \overline{B}_{l+1, \overline{\chi_1}}(cx) \overline{B}_{p-(l+1), \chi_2}(bx) dx \\ &= \chi_1(-1) \frac{k}{2} \left( \frac{k}{c} \right)^{p-1} \sum_{h=1}^{k-1} \sum_{j=1}^{k-1} \chi_1(h) \overline{\chi_2}(j) \overline{B}_p \left( \frac{cj}{k} + \frac{bh}{k} \right) \end{aligned}$$



for  $\gcd(b, c) = 1$ . In particular

$$\int_0^k x \overline{B}_{l+1, \overline{\chi_1}}(x) \overline{B}_{p-(l+1), \chi_2}(kx) dx = 0.$$

## 4 Integral of products of Bernoulli polynomials

### 4.1 Bernoulli polynomials

In [1], Agoh and Dilcher derived an explicit formula for the integral of product of three Bernoulli polynomials by considering the interval of integration  $[0, x]$ , rather than  $[0, 1]$  as in [8, 11, 18, 20, 21, 23], which are thus special cases of the following.

**Proposition 8** ([1, Proposition 3]) *For all  $l, m, n \geq 0$  we have*

$$\begin{aligned} & \frac{1}{l!m!n!} \int_0^x B_l(z) B_m(z) B_n(z) dz \\ &= \sum_{a=0}^{l+m} (-1)^a \sum_{j=0}^a \binom{a}{j} \frac{B_{l-a+j}(x) B_{m-j}(x) B_{n+a+1}(x) - B_{l-a+j} B_{m-j} B_{n+a+1}}{(l-a+j)! (m-j)! (n+a+1)!}. \end{aligned}$$

Recently, this has been extended by Hu et al [14] to integral of product of  $r$  Bernoulli polynomials. Let  $r$  be any positive integer. The multinomial coefficients  $\binom{\mu}{n_1, \dots, n_r}$  are defined by

$$\binom{\mu}{n_1, \dots, n_r} = \frac{\mu!}{n_1! \cdots n_r!},$$

where  $n_1 + \cdots + n_r = \mu$  and  $n_1, \dots, n_r \geq 0$ . Let

$$I_{n_1, \dots, n_r}(x) = \frac{1}{n_1! \cdots n_r!} \int_0^x B_{n_1}(z) \cdots B_{n_r}(z) dz, \quad (24)$$

$$C_{n_1, \dots, n_r}(x) = \frac{1}{n_1! \cdots n_r!} (B_{n_1}(x) \cdots B_{n_r}(x) - B_{n_1} \cdots B_{n_r}).$$

Hu et al [14] derive the following theorem.

**Theorem 9** ([14, Theorem 1.5]) *For any  $n_1, \dots, n_r \geq 0$ , we have*

$$I_{n_1, \dots, n_r}(x) = \sum_{a=0}^{n_1 + \cdots + n_{r-1}} (-1)^a \sum_{j_1 + \cdots + j_{r-1} = a} \binom{a}{j_1, \dots, j_{r-1}} C_{n_1 - j_1, \dots, n_{r-1} - j_{r-1}, n_r + a + 1}(x).$$

More recently, this result has been extended to the Appell polynomials (such polynomials satisfying  $\frac{d}{dz} A_n(z) = n A_{n-1}(z)$ ,  $n = 0, 1, 2, \dots$ ) by Liu et al [17, Theorem 1.1].

We can infer from (25) and (26) (see below) that these results are valid in a more general form and can be proved more easily. We take advantage of the following property of derivative

$$(f_1(z) \cdots f_m(z))^{(a)} = \sum_{j_1 + \cdots + j_m = a} \binom{a}{j_1, \dots, j_m} f_1^{(j_1)}(z) \cdots f_m^{(j_m)}(z), \quad (25)$$

which can be easily seen by induction on  $m$ .

Let  $f(z)$  be differentiable of order  $\mu + 1$ , and  $P_n(z)$  be a polynomial (or function) such that  $dP_n(z)/dz = nP_{n-1}(z)$ . Integration by parts gives

$$\frac{1}{n!} \int_0^x f(z) P_n(z) dz = \frac{1}{(n+1)!} [f(z) P_{n+1}(z)]_0^x - \frac{1}{(n+1)!} \int_0^x f'(z) P_{n+1}(z) dz.$$

Using  $\mu$  additional integrations by parts it is seen that

$$\begin{aligned} \frac{1}{n!} \int_0^x f(z) P_n(z) dz &= \sum_{a=0}^{\mu} \frac{(-1)^a}{(n+a+1)!} \left[ f^{(a)}(z) P_{n+a+1}(z) \right]_0^x \\ &\quad + \frac{(-1)^{\mu+1}}{(n+\mu+1)!} \int_0^x f^{(\mu+1)}(z) P_{n+\mu+1}(z) dz. \end{aligned} \quad (26)$$

In particular if  $f^{(\mu+1)}(z) = 0$ , then

$$\frac{1}{n!} \int_0^x f(z) P_n(z) dz = \sum_{a=0}^{\mu} \frac{(-1)^a}{(n+a+1)!} \left( f^{(a)}(x) P_{n+a+1}(x) - f^{(a)}(0) P_{n+a+1}(0) \right).$$

**Remark 10** By considering

$$f(z) = B_{n_1}(z) \cdots B_{n_{r-1}}(z) \text{ and } P_n(z) = B_{n_r}(z)$$

and for an Appell polynomial  $A_n(z)$

$$f(z) = A_{n_1}(z) \cdots A_{n_{r-1}}(z) \text{ and } P_n(z) = A_{n_r}(z)$$

in (26), and using (25) we have Hu et al's [14] and Liu et al's [17] results, respectively.

Let  $b_l$  ( $b_l \neq 0$ ) and  $y_l$  ( $1 \leq l \leq r$ ) be arbitrary real numbers and let

$$\begin{aligned} \widehat{I}_{n_1, \dots, n_r}(x; b; y) &= \widehat{I}_{n_1, \dots, n_r}(x; b_1, \dots, b_r; y_1, \dots, y_r) \\ &= \frac{1}{n_1! \cdots n_r!} \int_0^x B_{n_1}(b_1 z + y_1) \cdots B_{n_r}(b_r z + y_r) dz, \\ \widehat{C}_{n_1, \dots, n_r}(x; b; y) &= \widehat{C}_{n_1, \dots, n_r}(x; b_1, \dots, b_r; y_1, \dots, y_r) \\ &= \frac{1}{n_1! \cdots n_r!} \left( \prod_{l=1}^r B_{n_l}(b_l x + y_l) - \prod_{l=1}^r B_{n_l}(y_l) \right). \end{aligned}$$

We relate  $\widehat{I}_{n_1, \dots, n_r}(x; b; y)$  and  $\widehat{C}_{n_1, \dots, n_r}(x; b; y)$  in the following proposition.

**Proposition 11** For any  $n_1, \dots, n_r \geq 0$ , we have

$$\begin{aligned} \widehat{I}_{n_1, \dots, n_r}(x; b; y) &= \sum_{a=0}^{n_1 + \dots + n_{r-1}} (-1)^a \sum_{j_1 + \dots + j_{r-1} = a} \binom{a}{j_1, \dots, j_{r-1}} \\ &\quad \times b_1^{j_1} \cdots b_{r-1}^{j_{r-1}} b_r^{-a-1} \widehat{C}_{n_1-j_1, \dots, n_{r-1}-j_{r-1}, n_r+a+1}(x; b; y). \end{aligned} \quad (27)$$

**Proof.** Setting

$$f(z) = B_{n_1}(b_1 z + y_1) \cdots B_{n_{r-1}}(b_{r-1} z + y_{r-1}) \text{ and } P_n(z) = B_{n_r}(b_r z + y_r)$$

in (26) for  $\mu = n_1 + \cdots + n_{r-1}$ , then using (9) and (25) we have the desired result. ■

Similar to [1, Corollary 2] and [14, Corollary 1.7], it is seen from the definition of the integral  $\hat{I}_{n_1, \dots, n_r}(x; b; y)$  that the right-hand side of (27) is invariant under all permutations.

**Corollary 12** *Let  $T_{n_1, \dots, n_r}(x; b; y) = T_{n_1, \dots, n_r}(x; b_1, \dots, b_r; y_1, \dots, y_r)$  be the right-hand side of (27), and let  $\sigma \in S_r$ , where  $S_r$  is the symmetric group of degree  $r$ . Then for all  $n_1, \dots, n_r \geq 0$ ,*

$$T_{n_1, \dots, n_r}(x; b; y) = T_{\sigma(n_1), \dots, \sigma(n_r)}(x; b_\sigma; y_\sigma),$$

where  $b_\sigma = (b_{\sigma(n_1)}, \dots, b_{\sigma(n_r)})$  and  $y_\sigma = (y_{\sigma(n_1)}, \dots, y_{\sigma(n_r)})$ .

**Proof of Corollary 4.** From Corollary 12 and Proposition 11 for  $r = 2$ , we have

$$\begin{aligned} & \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} (B_{n-a}(b_1 x + y_1) B_{m+a+1}(b_2 x + y_2) - B_{n-a}(y_1) B_{m+a+1}(y_2)) \\ &= \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} (B_{m-a}(b_2 x + y_2) B_{n+a+1}(b_1 x + y_1) - B_{m-a}(y_2) B_{n+a+1}(y_1)). \end{aligned} \quad (28)$$

Let

$$\begin{aligned} T &:= \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} B_{n-a}(y_1) B_{m+a+1}(y_2) \\ &\quad - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} B_{m-a}(y_2) B_{n+a+1}(y_1). \end{aligned}$$

This may be written as

$$\begin{aligned} T &= \sum_{a=0}^n (-1)^{n-a} \binom{m+n+1}{a} b_1^{n-a} b_2^{a-n-1} B_a(y_1) B_{m+n+1-a}(y_2) \\ &\quad - \sum_{a=0}^m (-1)^{m-a} \binom{m+n+1}{a} b_2^{m-a} b_1^{a-m-1} B_a(y_2) B_{m+n+1-a}(y_1). \end{aligned}$$

Without loss of generality we may assume that  $n \geq m$ ; in this case we divide the first sum into two parts, from 0 to  $m$  and  $m+1$  to  $n$ , and the results is

$$\begin{aligned} & \sum_{a=0}^m (-1)^{n-a} \binom{m+n+1}{a} b_1^{n-a} b_2^{a-n-1} B_a(y_1) B_{m+n+1-a}(y_2) \\ &= \sum_{a=n+1}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} b_1^{a-m-1} b_2^{m-a} B_{m+n+1-a}(y_1) B_a(y_2) \end{aligned}$$

and

$$\begin{aligned} & \sum_{a=m+1}^n (-1)^{n-a} \binom{m+n+1}{a} b_1^{n-a} b_2^{a-n-1} B_a(y_1) B_{m+n+1-a}(y_2) \\ &= \sum_{a=m+1}^n (-1)^{m+1-a} \binom{m+n+1}{a} b_1^{a-m-1} b_2^{m-a} B_{m+n+1-a}(y_1) B_a(y_2) \end{aligned}$$

(for  $m = n$  the above sum vanishes). Thus,

$$T = \frac{1}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} B_{m+n+1-a}(y_1) B_a(y_2). \quad (29)$$

So, combining (28) and (29), the desired result follows. ■

Observe that starting from the left-hand side of (6) and proceeding as in the proof of (29), the right-hand side of (6) turns into

$$\frac{1}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} B_{m+n+1-a}(b_1 x + y_1) B_a(b_2 x + y_2),$$

which implies that

$$\begin{aligned} & \sum_{a=0}^{m+n+1} (-1)^a \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} B_{m+n+1-a}(b_1 x + y_1) B_a(b_2 x + y_2) \\ &= \sum_{a=0}^{m+n+1} (-1)^a \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} B_{m+n+1-a}(y_1) B_a(y_2) \end{aligned}$$

holds for all  $x$ . Note that obtaining the relation above is not so clear without using the integrals for the case  $b_1 \neq \pm b_2$ .

Now we set  $b_1 = b_2 = 1$  in (6). Then, by the fact  $B_m(1-x) = (-1)^m B_m(x)$ ,

$$T = (-1)^n \sum_{a=0}^{m+n+1} \binom{m+n+1}{a} B_{m+n+1-a}(1-y_1) B_a(y_2).$$

We take  $x = 1 - y_1$ ,  $y = y_2$  and  $p = m + n + 1$  in the well-known identity

$$\sum_{a=0}^p \binom{p}{a} B_{p-a}(x) B_a(y) = p(x+y-1) B_{p-1}(x+y) - (p-1) B_p(x+y) \quad (30)$$

and we find that

$$\begin{aligned} & \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} B_{n-a}(x+y_1) B_{m+a+1}(x+y_2) \\ & - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} B_{m-a}(x+y_2) B_{n+a+1}(x+y_1) \\ &= (-1)^m (m+n+1) (y_2 - y_1) B_{m+n}(y_1 - y_2) + (-1)^m (m+n) B_{m+n+1}(y_1 - y_2). \end{aligned} \quad (31)$$

Note that (31) reduces to [1, Proposition 2] for  $y_1 = y_2$ . The case  $b_1 = -b_2 = -1$  in (6) is equivalent to (31) which can be seen by setting  $1 - y_1$  instead of  $y_1$  and using  $B_n(1-y) = (-1)^n B_n(y)$ .

We mention also that the right-hand side of (27) becomes simple for real numbers  $y_l$ ,  $b_l = (1 - 2y_l)/q$ , ( $y_l \neq 1/2$ ),  $1 \leq l \leq r$ , and  $x = q \neq 0$ . Then

$$\begin{aligned} & \hat{I}_{n_1, \dots, n_r} \left( q; \frac{1-2y_1}{q}, \dots, \frac{1-2y_r}{q}; y_1, \dots, y_r \right) \\ &= \sum_{a=0}^{n_1 + \dots + n_{r-1}} (-1)^a \sum_{j_1 + \dots + j_{r-1} = a} \binom{a}{j_1, \dots, j_{r-1}} b_1^{j_1} \dots b_{r-1}^{j_{r-1}} b_r^{-a-1} \\ & \times \frac{((-1)^{n_1 + \dots + n_r + 1} - 1)}{(n_1 - j_1)! \dots (n_r + a + 1)!} B_{n_1 - j_1}(y_1) \dots B_{n_{r-1} - j_{r-1}}(y_{r-1}) B_{n_r + a + 1}(y_r) \end{aligned}$$

since  $B_{n_l-j_l}(b_l q - y_l) = B_{n_l-j_l}(1 - y_l) = (-1)^{n_l-j_l} B_{n_l-j_l}(y_l)$  and  $j_1 + \dots + j_{r-1} = a$ . Therefore if  $n_1 + \dots + n_r + 1$  is even, then

$$\hat{I}_{n_1, \dots, n_r} \left( q; \frac{1-2y_1}{q}, \dots, \frac{1-2y_r}{q}; y_1, \dots, y_r \right) = 0, \quad (32)$$

and if  $n_1 + \dots + n_r + 1$  is odd, then

$$\begin{aligned} & \hat{I}_{n_1, \dots, n_r} \left( q; \frac{1-2y_1}{q}, \dots, \frac{1-2y_r}{q}; y_1, \dots, y_r \right) \\ &= -2q \sum_{a=0}^{n_1+\dots+n_{r-1}} (-1)^a \frac{(1-2y_r)^{-a-1}}{(n_r+a+1)!} B_{n_r+a+1}(y_r) \\ & \quad \times \sum_{j_1+\dots+j_{r-1}=a} \binom{a}{j_1, \dots, j_{r-1}} \prod_{l=1}^{r-1} \frac{(1-2y_l)^{j_l}}{(n_l-j_l)!} B_{n_l-j_l}(y_l). \end{aligned} \quad (33)$$

For example, we have

$$\begin{aligned} & \int_0^1 B_3(-z+1) B_4(3z-1) B_{16}(5z-2) dz = 0, \\ & \frac{1}{3!4!15!} \int_0^1 B_3(-z+1) B_4(3z-1) B_{15}(-3z+2) dz \\ &= -2 \sum_{a=0}^7 \frac{B_{16+a}(2)}{(16+a)!} \sum_{i=0}^a \binom{a}{i} 3^{-i-1} \frac{B_{3-i} B_{4-a+i}(-1)}{(3-i)!(4-a+i)!}. \end{aligned}$$

## 4.2 Generalized Bernoulli polynomials

The counterpart of Proposition 11 for generalized Bernoulli polynomials  $B_{n_l, \chi_l}(b_l z + y_l)$  can be also expressed for non-principal primitive characters  $\chi_l$  of modulus  $k_l$ ,  $1 \leq l \leq r$ . But in this case,  $n_1, \dots, n_r$  must be  $\geq 1$  since the degree of  $B_{n, \chi}(z)$  is not greater than  $n-1$ . Moreover, the summation from  $a=0$  to  $n_1 + \dots + n_{r-1}$  on the right-hand side of (27) can be replaced by the summation from  $a=0$  to  $n_1 + \dots + n_{r-1} - (r-1)$ . Under this circumstances the analogues of (32) and (33) are valid according to  $\chi_1(-1) \dots \chi_r(-1) (-1)^{n_1+\dots+n_r+1} = 1$  or  $-1$ , but in this case  $b_l = -2y_l/q$ , ( $y_l \neq 0$ ),  $1 \leq l \leq r$ . Furthermore, first writing the analogue of (28) then adopting the arguments in the proof of (29), it can be seen for all  $m, n \geq 1$  that

$$\begin{aligned} & \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} B_{n-a, \chi_1}(b_1 x + y_1) B_{m+a+1, \chi_2}(b_2 x + y_2) \\ & \quad - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} B_{m-a, \chi_2}(b_2 x + y_2) B_{n+a+1, \chi_1}(b_1 x + y_1) \\ &= \frac{(-1)^{m+1}}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^a \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} B_{m+n+1-a, \chi_1}(y_1) B_{a, \chi_2}(y_2). \end{aligned} \quad (34)$$

We notice that if  $(-1)^{m+n} \chi_1(-1) \chi_2(-1) = 1$  and  $y_1 = y_2 = 0$ , then the right-hand side of (34) vanishes, since  $B_{m+n+1-a, \chi_1} B_{a, \chi_2} = 0$  by (11). In the case  $(-1)^{m+n} \chi_1(-1) \chi_2(-1) = -1$  and  $y_1 = y_2 = 0$  the sum on the right-hand side of (34) is closely related to the reciprocity formulas of character Dedekind sums. For

instance, from Corollary 3 and (34), we have

$$\begin{aligned}
& (p+1) \left( bc^p \widehat{S}_p(b, c : \overline{\chi_1}, \chi_2) + cb^p \widehat{S}_p(c, b : \overline{\chi_2}, \chi_1) \right) \\
&= \sum_{a=0}^{n-1} \binom{m+n+1}{n-a} c^{m+a+1} b^{n-a} B_{n-a, \chi_1}(-cx) B_{m+a+1, \chi_2}(bx) \\
&\quad + \sum_{a=0}^{m-1} \binom{m+n+1}{m-a} b^{n+a+1} c^{m-a} B_{m-a, \chi_2}(bx) B_{n+a+1, \chi_1}(-cx) \\
&= \sum_{j=0}^{p+1} \binom{p+1}{j} c^j b^{p+1-j} B_{p+1-j, \chi_1} B_{j, \chi_2}
\end{aligned}$$

for  $(-1)^{p+1} \chi_1(-1) \chi_2(-1) = 1$  and  $p = m + n$  ( $m, n \geq 1$ ).

### 4.3 Laplace transform of Bernoulli function

Let  $\operatorname{Re}(s) > 0$  and  $|s/t| < 2\pi$ . Setting  $f(u) = e^{-su}$  and  $P_n(u) = \overline{B}_n(tu + y)$ ,  $n \geq 1$ , in (26) gives

$$\begin{aligned}
\frac{1}{n!} \int_0^x e^{-su} \overline{B}_n(tu + y) du &= \sum_{a=0}^{\mu} \frac{s^a t^{-a-1}}{(n+a+1)!} \{e^{-sx} \overline{B}_{n+a+1}(tx + y) - \overline{B}_{n+a+1}(y)\} \\
&\quad + \left(\frac{s}{t}\right)^{\mu+1} \frac{1}{(n+\mu+1)!} \int_0^x e^{-su} \overline{B}_{n+\mu+1}(tu + y) du.
\end{aligned} \tag{35}$$

Since the function  $\overline{B}_m(u) = B_m(u - [u])$  is bounded, the integrals in (35) converge absolutely and  $e^{-sx} \overline{B}_{n+a+1}(tx + y)$  tends to 0 as  $x \rightarrow \infty$ . Then, letting  $x \rightarrow \infty$ , we have

$$\begin{aligned}
\frac{1}{n!} \int_0^{\infty} e^{-su} \overline{B}_n(tu + y) du &= -\frac{t^n}{s^{n+1}} \sum_{a=0}^{\mu} \frac{\overline{B}_{n+a+1}(y)}{(n+a+1)!} \frac{s^{n+a+1}}{t^{n+a+1}} \\
&\quad + \left(\frac{s}{t}\right)^{\mu+1} \frac{1}{(n+\mu+1)!} \int_0^{\infty} e^{-su} \overline{B}_{n+\mu+1}(tu + y) du.
\end{aligned} \tag{36}$$

From (7) the sum in (36) converges absolutely for  $|s/t| < 2\pi$  as  $\mu \rightarrow \infty$ . Also the sequence of the functions  $g_{\mu}(u) = s^{\mu} \overline{B}_{\mu}(tu + y) / \mu! t^{\mu}$  converges uniformly to 0 for  $|s/t| < 2\pi$ . Thus, letting  $\mu \rightarrow \infty$  we find that

$$\frac{1}{n!} \int_0^{\infty} e^{-su} \overline{B}_n(tu + y) du = -\frac{t^n}{s^{n+1}} \sum_{a=n+1}^{\infty} \frac{\overline{B}_a(y)}{a!} \frac{s^a}{t^a}. \tag{37}$$

Since  $\overline{B}_a(y) = B_a(\{y\})$  for  $a \geq 2$ , where  $\{y\} = y - [y]$ , we have

$$\sum_{a=n+1}^{\infty} \frac{\overline{B}_a(y)}{a!} u^a = -\sum_{a=0}^n \frac{B_a(\{y\})}{a!} u^a + \frac{ue^{\{y\}u}}{e^u - 1}$$

by (7). Therefore, we arrive at the Laplace transform of  $\overline{B}_n(tu + y)$

$$\int_0^{\infty} e^{-su} \overline{B}_n(tu + y) du = n! \frac{t^n}{s^{n+1}} \left( \sum_{a=0}^n \frac{B_a(\{y\})}{a!} \frac{s^a}{t^a} - \frac{s}{t} \frac{e^{\{y\}s/t}}{e^{s/t} - 1} \right) \tag{38}$$

for all  $Re(s) > 0$  and  $n \geq 1$ , by analytic continuation. Differentiating  $r$  times both sides of (38) with respect to  $s$ , then using the well-known identity  $\sum_{r=0}^m \binom{m}{r} B_{m-r} x^r = B_m(x)$  we deduce that

$$\begin{aligned} & \int_0^\infty e^{-su} B_m(u) \overline{B}_n(u) du \\ &= \sum_{r=0}^m \binom{m}{r} B_{m-r} \left( \sum_{a=0}^n \binom{n}{a} \frac{(n+r-a)!}{s^{n+1+r-a}} B_a - n! (-1)^r \frac{d^r}{ds^r} \frac{s^{-n}}{e^s - 1} \right). \end{aligned}$$

By the similar way, we have

$$\frac{1}{n!} \int_0^\infty e^{-su} \overline{B}_{n,\chi}(tu) du = \frac{1}{s} \sum_{a=0}^n \frac{B_{a,\chi}}{a!} \left( \frac{t}{s} \right)^{n-a} - \frac{t^{n-1}}{s^n} \sum_{j=0}^{k-1} \frac{\overline{\chi}(j) e^{js/t}}{e^{ks/t} - 1}$$

for  $Re(z) > 0$ ,  $n \geq 1$ , and a non-principal primitive character  $\chi$  of modulus  $k$ .

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